Quantum Mechanics on the Noncommutative Torus

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Abstract

We analyze the algebra of observables of a charged particle on a noncommutative torus in a constant magnetic field. We present a set of generators of this algebra which coincide with the generators for a commutative torus but at a different value of the magnetic field, and demonstrate the existence of a critical value of the magnetic field for which the algebra reduces. We then obtain the irreducible representations of the algebra and relate them to noncommutative bundles. Finally we comment on Landau levels, density of states and the critical case.

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1 Introduction

Noncommutative spaces arise as solutions of matrix models and in the effective description of branes in string theory [1]-[6]. The fluctuations of these brane or the matrix model solutions are described by noncommutative field theories. It is expected that quanta of these field theories will represent particles moving on the underlying noncommutative spaces. It is therefore of interest to examine the dynamics of these quantum mechanical particles.

Although noncommutative field theories have been extensively studied, the corresponding quantum mechanical problem has received relatively little attention until recently [7]-[15]. Some related studies of finite quantum mechanics and its relation to the noncommutative torus are [16]-[18], and for a discussion of the modular invariace see [19][20]. In [15], in particular, the problem of a quantum particle moving on a noncommutative plane and sphere was examined and solved. The results revealed a Landau level picture analogous to the commutative case. An important qualitative difference, however, was a modified density of states and the existence of a critical value of the magnetic field at which this density diverges. This may have a relevance to the recently proposed analogy between noncommutative field theory and the quantum Hall effect [21].

The purpose of this paper is to analyze the Landau problem for the case of a flat periodic space, that is, a two-torus. This situation is interesting even in the commutative case, being closely related to the Hofstadter problem. As we will show, a mapping can be established between the commutative and noncommutative cases, revealing the features of the model and the emergence of the critical magnetic field.

We will follow the approach of identifying the algebra of physical observables for the model and finding its irreducible representations. This is conceptually more fundamental than corresponding treatments based on explicit wave equations and bypasses the questions of extended gauge invariance, single-valuedness of the wavefunction etc. It is similar in spirit to the approach of isolating and examining only the gauge invariant observables and

their algebra, rather than working with a gauge theory and imposing gauge invariance on the states. A connection with more traditional approaches will also be given.

The structure of the paper is as follows. In Section 2 we write the algebra of observables on the noncommutative torus and establish its equivalence with a modified commutative torus. In Section 3 we use this equivalence and obtain irreducible representations of the algebra. In Section 4 we review the algebra of observables and its representations for the noncommutative plane. In Section 5 we recover the torus representations through a reduction of the planar representation and show that they correspond to quantum bundles over the noncommutative torus. In the last section we discuss the energy eigenstates (Landau levels) and the density of states and comment on the critical case.

2 Algebra of observables

A noncommutative plane is defined in terms of two flat noncommutative coordinates X_1, X_2 obeying the commutation relation

$$[X_1, X_2] = i\Theta \ , \tag{1}$$

with Θ a c-number parameter, equal to the noncommutative length scale squared. The motion of a particle on such a plane will be described by the above noncommutative coordinates and two corresponding momenta P_i, defined as shift operators on the X_i. In the presence of a constant magnetic field $\mathcal{B}$ the commutator of the P_i does not vanish and becomes proportional to $\mathcal{B}$. (P_i are the 'gauge invariant' or 'kinematical' momenta, rather than the 'canonical' momenta.) The complete algebra of observables is, then

$$\begin{cases} [\, X_1, X_2 \,] & = & i\Theta \ , \\ [\, X_i, P_j \,] & = & i\delta_{ij} \ , \\ [\, P_1, P_2 \,] & = & i\mathcal{B} \ . \end{cases} \tag{2}$$

The Hamiltonian is the free one

$$H = \frac{1}{2} \left(P_1^2 + P_2^2 \right) . \tag{3}$$

To describe a particle on a noncommutative torus, we further impose the periodicity condition that

$$\vec{X} \sim \vec{X} + \vec{a}_i , \quad i = 1, 2 , \tag{4}$$

where the components of $\vec{a}_i$ are c-numbers. This represents an oblique torus with period vectors $\vec{a}_1, \vec{a}_2$ and area $A = \vec{a}_1 \times \vec{a}_2 = a_{11}a_{22} - a_{12}a_{21}$.

Clearly the X_i are not physical operators, since they are not uniquely fixed by the position of the particle on the torus. As physical operators we take the exponentials

$$U_i = e^{i \vec{b}_i \cdot \vec{X}} ,$$

where $\vec{b}_i$ are the dual torus vectors satisfying

$$\vec{a}_i \cdot \vec{b}_j = 2\pi \delta_{ij} .$$

The above unitary U_i are invariant under the shifts (4) and are, therefore, physical operators. The complete set of physical observables for this particle are the two unitary 'position' operators U_i and the two hermitian momenta P_i satisfying the commutation relations

$$\begin{cases} U_1 U_2 & = & U_2 U_1 \, e^{-i\theta} , \\ P_i \, U_j & = & U_j \left(P_i + a_{ji} \right) , \\ [\, P_1, P_2 \,] & = & i \, \mathcal{B} , \end{cases} \tag{5}$$

with $\theta = (2\pi)^2 \Theta / A$ a dimensionless parameter ($1/\theta$ effectively counts the noncommutative area 'quanta' contained in the torus). Finally, we can cast (5) in a form not explicitly involving the period vectors by defining new momenta

$$p_i = \frac{1}{2\pi} \vec{a}_i \cdot \vec{P} ,$$

in terms of which we obtain

$$\begin{cases} U_1 U_2 & = & U_2 U_1 \, e^{-i\theta} , \\ p_i \, U_j & = & U_j \left(p_i + \delta_{ij} \right) , \\ [\, p_1, p_2 \,] & = & i \, B , \end{cases} \tag{6}$$

where
$$B = \frac{A\mathcal{B}}{(2\pi)^2} \; .$$

The associative algebra generated by U_i and p_i and satisfying the relations (6) will be denoted by $\mathcal{T}_{\theta,B}$. We can also introduce $x_i = \vec{b}_i \cdot \vec{X}$ and rewrite (2) as

$$\begin{cases} [\,x_1, x_2\,] & = & i\theta \; , \\ [\,x_i, p_j\,] & = & i\delta_{ij} \; , \\ [\,p_1, p_2\,] & = & i\,B \; . \end{cases} \tag{7}$$

The associative algebra generated by x_i and p_i and satisfying the relations (7) will be denoted by $\mathcal{P}_{\theta,B}$. Note that the torus algebra $\mathcal{T}_{\theta,B}$ is a subalgebra of the plane algebra $\mathcal{P}_{\theta,B}$. The Hamiltonian is given by

$$H = \frac{1}{2}\,(P_1^2 + P_2^2) = \frac{b_1^2}{2}p_1^2 + \frac{b_2^2}{2}p_2^2 + \vec{b}_1 \cdot \vec{b}_2(p_1 p_2 + p_2 p_1) \; .$$

We will now show that $\mathcal{T}_{\theta,B}$ is isomorphic to $\mathcal{T}_{0,\tilde{B}}$ where $\tilde{B} = B/(1 - B\theta)$. First let

$$\tilde{U}_1 \equiv U_1 e^{i\alpha p_2} \; , \; \tilde{U}_2 \equiv U_2 e^{i\beta p_1} \; ,$$

where α and β are two real c-numbers. Then the $\tilde{U}_i$'s commute if

$$\alpha\beta B + \alpha - \beta - \theta = 0 \; . \tag{8}$$

One can also show that

$$\begin{cases} p_1 \tilde{U}_1 & = & \tilde{U}_1(p_1 + 1 - \alpha B) \; , \\ p_2 \tilde{U}_2 & = & \tilde{U}_2(p_2 + 1 + \beta B) \; . \end{cases}$$

If we set $1 - \alpha B = \pm(1 + \beta B)$ and define $\tilde{p}_1 = p_1/(1 - \alpha B)$ and $\tilde{p}_2 = \pm p_2/(1 - \alpha B)$ we have

$$\begin{cases} \left[\tilde{U}_1, \tilde{U}_2\right] & = & 0 \; , \\ \tilde{p}_i \tilde{U}_j & = & \tilde{U}_j(\tilde{p}_i + \delta_{ij}) \; , \\ [\tilde{p}_1, \tilde{p}_2] & = & i\tilde{B} \; , \end{cases}$$

where
$$\tilde{B} = \pm\frac{B}{(1 - \alpha B)^2} \; .$$

Using (8) one can show that $(1 - \alpha B)^2 = \pm(1 - \theta B)$, thus the choice of sign is dictated by the sign of $1 - \theta B = 1 - \Theta \mathcal{B}$. Then $\tilde{U}_i$ and $\tilde{p}_i$ generate the algebra of observable $\mathcal{T}_{0,\tilde{B}}$ of a charged particle in a magnetic field $\tilde{B}$ on a commutative torus. In terms of $\tilde{p}_i$ the Hamiltonian takes the form

$$H = \frac{|1 - \Theta\mathcal{B}|}{2} \left\{ \frac{b_1^2}{2}\tilde{p}_1^2 + \frac{b_2^2}{2}\tilde{p}_2^2 + \vec{b}_1 \cdot \vec{b}_2(\tilde{p}_1\tilde{p}_2 + \tilde{p}_2\tilde{p}_1) \right\} \ ,$$

thus it only differs from the standard Hamiltonian on the commutative torus by an overall normalization.

When $1 - \Theta\mathcal{B}$ vanishes we cannot define $\tilde{p}_i$ as above. For the choice $\alpha = -\beta = \theta$, however, we note that each p_i commutes with the $\tilde{U}_i$'s which mutually commute (this was not possible before). Thus for the critical value $\mathcal{B} = \Theta^{-1}$ the algebra reduces into the direct product of a Heisenberg algebra and two commuting U(1) operators and becomes

$$\left\{ \begin{array}{rcl} \tilde{U}_1\tilde{U}_2 & = & \tilde{U}_2\tilde{U}_1 \ , \\ \left[p_i, \tilde{U}_j\right] & = & 0 \ , \\ \left[p_1, p_2\right] & = & i\,B \ . \end{array} \right. \tag{9}$$

This signals the reduction of the Hilbert space at criticality.

In the next sections we use two different methods to study the representations of the algebra $\mathcal{T}_{\theta,B}$, first using the equivalence to the commutative torus, and then obtaining the quantum bundles over the noncommutative torus.

3   Irreducible representations

Since the noncommutative algebra (away from criticality) is equivalent to the commutative one with a new value of the magnetic field $\tilde{B}$, it suffices to study the irreducible representations of the algebra (6) at $\theta = 0$. We need the Casimirs of the algebra. First note that the operators

$$\left\{ \begin{array}{rcl} W_1 & = & \exp(-i\frac{m_1}{B}\tilde{p}_1)\tilde{U}_2{}^{m_1} \ , \\ W_2 & = & \exp(-i\frac{m_2}{B}\tilde{p}_2)\tilde{U}_1{}^{-m_2} \ , \end{array} \right.$$

where m_1 and m_2 are arbitrary integers, commute with $\tilde{p}_i$. They will also commute with $\tilde{U}_i$ if

$$2\pi \tilde{B} = m/n \tag{10}$$

for some integers m and $n > 0$, and we take both m_1 and m_2 to be multiples of m. If we take (m, n) to be relatively prime integers, we obtain the minimal operators W_i, forming a complete set of generators of the Casimirs of the algebra, by choosing $m_1 = m_2 = m$. So we have

$$\begin{cases} W_1 &= e^{-2\pi i n \tilde{p}_1} \tilde{U}_2{}^m \ , \\ W_2 &= e^{-2\pi i n \tilde{p}_2} \tilde{U}_1^{-m} \ . \end{cases}$$

For $n = 1$, (10) is the familiar condition for an integer number of magnetic flux quanta through the surface of the torus, but with the modified magnetic field $\tilde{B}$ now entering the quantization condition. The representation of the algebra of observables in that case is rather straightforward. As we will see, for $n > 1$ one can obtain a representation by either considering an enlarged torus of area nA or equivalently by introducing an internal quantum number corresponding to an n-fold wave function.

Next we find a complete set of commuting operators for the algebra containing $\tilde{U}_i$. The most general choice (up to some $\tilde{U}_i$ factors) is to add the following two operators

$$Z_i = e^{-2\pi i \vec{N}_i \cdot \vec{\tilde{p}}} \ , \ i = 1, 2$$

defined in terms of two arbitrary integral vectors $\vec{N}_1$ and $\vec{N}_2$ satisfying $n = \vec{N}_1 \times \vec{N}_1 = N_{11}N_{22} - N_{12}N_{21}$. (Note that $\vec{N}_1, \vec{N}_2$ define an n-fold enlarged torus.) The set $\{\tilde{U}_1, \tilde{U}_2, Z_1, Z_2\}$ is complete and in particular the Casimir operators W_i can be written in terms of the elements of this set

$$\begin{cases} W_1 &= e^{\pi i m N_{22} N_{12}} Z_1^{N_{22}} Z_2^{-N_{12}} \tilde{U}_2{}^m \ , \\ W_2 &= e^{\pi i m N_{11} N_{21}} Z_1^{-N_{21}} Z_2^{N_{11}} \tilde{U}_1^{-m} \ . \end{cases} \tag{11}$$

Then we can find a state denoted $|\vec{0}\rangle$ which satisfies[a]

$$
\begin{cases}
\tilde{U}_i|\vec{0}\rangle &= |\vec{0}\rangle \;, \\
Z_i|\vec{0}\rangle &= e^{i\zeta_i}|\vec{0}\rangle \;.
\end{cases}
\tag{12}
$$

An irreducible representation is obtained by acting on this state with operators which do not commute with $\tilde{U}_i$ or Z_i and modding out by zero norm states. The following states

$$
|\vec{\phi}\rangle = e^{-i\vec{\phi}\cdot\vec{p}}|\vec{0}\rangle
\tag{13}
$$

are obtained in this way and satisfy

$$
\begin{cases}
\tilde{U}_i|\vec{\phi}\rangle &= e^{i\phi_i}|\vec{\phi}\rangle \;, \\
Z_i|\vec{\phi}\rangle &= e^{i(\zeta_i - \frac{m}{n}\vec{N}_i\times\vec{\phi})}|\vec{\phi}\rangle \;.
\end{cases}
$$

Note that the states $|\vec{\phi}\rangle$ and $|\vec{\phi}'\rangle$ have the same U_i and Z_i eigenvalues if

$$
\vec{\phi}' = \vec{\phi} + 2\pi\vec{N}_j \;,
$$

and need not be linearly independent. In fact, it is consistent to identify them, as

$$
|\vec{\phi} + 2\pi\vec{N}_i\rangle = e^{i(\zeta_i - \frac{m}{2n}\vec{N}_i\times\vec{\phi})}|\vec{\phi}\rangle \;.
\tag{14}
$$

In other words, the difference of the above two states is a null state and can be consistently set to zero. The independent states are thus labeled by the vector $\vec{\phi}$ living in a fundamental cell of the enlarged lattice generated by $\vec{N}_i$ and they form an irreducible representation. Using (11) we obtain the following expressions for the Casimirs $W_i = e^{i\omega_i}$ in this representation:

$$
\omega_i = M_{ij}\,\zeta_j + \pi l_j \;,
$$

where $l_i = mM_{i1}M_{i2}\mathrm{mod}(2)$ (no i summation) and M is the integral matrix satisfying $MN = n\mathbb{1}$. We note that the Casimirs of the representation are related to the Wilson lines around the periods of the torus.

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[a]We could start with an arbitrary U_i eigenstate and use exponentials of p_i to get the state $|\vec{0}\rangle$ which has $\tilde{U}_i$ eigenvalues 1.

We have shown that for every choice of an integral lattice of area n and of phases ζ_i, we can construct in a standard way an irreducible representation of the algebra $\mathcal{T}_{0,\tilde{B}}$ starting from an anchor state satisfying (12). We will now show that any two such representations (with different $\vec{N}_i$ and ζ_i) having the same Casimirs are equivalent.

First we give an abstract proof. Consider two representations: the first obtained using $\vec{N}_i$ and ζ_i and the second obtained using $\vec{N}'_i$ and ζ'_i. Since the operators Z_i also act in the second representation and they commute with $\tilde{U}_i$ they can be simultaneously diagonalized. Taking an arbitrary eigenvector and acting on it with translation operators we can obtain an eigenvector with unit $\tilde{U}_i$ eigenvalues. The result is also an eigenvector of Z_i with some eigenvalues $e^{i\zeta_i}$. Thus we have found an anchor state of the first representation as a state in the second representation and thus the two representations are equivalent.

We can also work out the explicit map between states in two representations. First we find an anchor state of the first representation (denoted with unprimed states) as a state in the second representation (denoted with primed states). Since Z_i are lattice translation operators we make the following ansatz:

$$|\vec{0}\rangle = \sum_{k_1,k_2=0}^{n-1} C_{k_1,k_2}|2\pi k_i \vec{N}_i\rangle' \ . \tag{15}$$

On the right hand side the states are in the primed representation but the sum is over lattice points of the unprimed representation. Assuming that we can shift the summation index the eigenvalue equations for Z_i give the following equations for the coefficients

$$\begin{cases} C_{k_1+1,k_2} &= e^{-i\zeta_1}\, e^{-\pi i m k_2} C_{k_1,k_2} \ , \\ C_{k_1,k_2+1} &= e^{-i\zeta_2}\, e^{+\pi i m k_1} C_{k_1,k_2} \ . \end{cases} \tag{16}$$

These are solved by

$$C_{k_1 k_2} = e^{-i\vec{k}\cdot\vec{\zeta}}\, e^{\pi i m k_1 k_2} \ ,$$

for which one can check that we have a periodic summand in (15) if the phases ζ_i and ζ'_i and the two lattices are such that they give the same Casimirs (11).

By acting with $e^{-i\vec{\phi}\cdot\vec{p}}$ on the anchor state we obtain

$$|\vec{\phi}\rangle = \sum_{k_1,k_2=0}^{n-1} e^{-i\vec{k}\cdot\vec{\zeta}} e^{\pi i m k_1 k_2} e^{i\frac{m}{2n}k_i \vec{N}_i \times \vec{\phi}} |\vec{\phi} + 2\pi k_i \vec{N}_i\rangle' \ .$$

which establishes the complete explicit mapping between the two representations.

The above concludes the derivation of the irreducible Hilbert space for the particle on the torus. An important point is that the position operators $\tilde{U}_i$ do not suffice to fully characterize the states; the additional operators Z_i are also required. (They are only absent in the case of integer quantization of $\tilde{B}$, that is $n = 1$, in which case the Z_i can be expressed entirely as functions of the $\tilde{U}_i$ and the Casimirs W_i.) This means that the set of $\tilde{U}_i$ eigenstates alone is not complete and a 'torus wavefunction' description of the states in terms of (quasi-) periodic functions on the torus is inadequate. This is common to both the commutative and noncommutative case. To fully specify the state an additional set of discrete degrees of freedom are needed. Indeed, looking at (14) we see that $\vec{\phi}$, which labels independent states, takes values on an n-fold enlarged torus. Each point on the fundamental torus has n images on the enlarged torus and we need to know the value of the wavefunction on each of these images to fully specify the state. This amounts to promoting the $\tilde{U}_i$-wavefunction into an n-component vector.

To make this more explicit, let us consider the representation defined in terms of the enlarged lattice $\vec{N}_1 = (n,0)$, $\vec{N}_2 = (0,1)$. This corresponds to enlarging the fundamental torus n-fold in the 1-direction. There is no loss of generality since representations defined in terms of any $\vec{N}_i$ are equivalent. Then define the states

$$|\vec{\phi}; q\rangle\!\rangle = e^{-\frac{i}{n}(\zeta_1 - \frac{m}{2n}\phi_2)q} |\phi_1 + 2\pi q, \phi_2\rangle \ , \quad q = 0, \ldots n - 1 \ .$$

For each $\vec{\phi}$ the above states form an n-vector with components labeled by q. By virtue of (14) this vector is quasiperiodic on the fundamental torus for the variable $\vec{\phi}$, namely

$$\begin{cases} |\phi_1 + 2\pi, \phi_2; q\rangle\!\rangle &= \quad e^{\frac{i}{n}(\zeta_1 - \frac{m}{2n}\phi_2)} |\phi_1, \phi_2; q + 1\rangle\!\rangle \ , \\ |\phi_1, \phi_2 + 2\pi; q\rangle\!\rangle &= \quad e^{i(\zeta_2 + \frac{m}{2n}(\phi_1 + 2\pi q))} |\phi_1, \phi_2; q + 1\rangle\!\rangle \ . \end{cases}$$

We see that shifts in ϕ_1 and ϕ_2 act as "shift" and "clock" matrices on the q components (modulo $\vec{\phi}$-dependent phases). So, overall, the large gauge transformations associated with shifts in the fundamental torus for $\vec{\phi}$ have been promoted to $U(n)$ nonabelian transformations.

4   Particle on the Quantum Plane in B-field

An alternative way of obtaining irreducible representation of the algebra of observables (6) of the charged particle on the quantum *torus* is to start with representations of the algebra of observables (7) of the particle on the quantum *plane*. These decompose as direct sums of irreducible representations of the algebra (6). In this section we present two ways of obtaining representations of the algebra of observable in the quantum plane case (7), and then show that they are equivalent. In the next section we show how to select a particular irreducible representation of the algebra (6).

The first method is similar to what we did in the previous chapter for the quantum torus. First we introduce a new set of generators

$$\left\{ \begin{array}{rcl} \tilde{x}_i & = & x_i - \kappa\varepsilon_{ij}p_j \ , \\ \tilde{p}_i & = & \frac{p_i}{1+\kappa B} \ , \end{array} \right.$$

where $x_i = \vec{b}_i \cdot \vec{X}$ and

$$\kappa = \frac{(1-\theta B)^{1/2} - 1}{B} \ .$$

They satisfy

$$\left\{ \begin{array}{rcl} [\tilde{x}_1, \tilde{x}_2] & = & 0 \ , \\ [\tilde{x}_i, \tilde{p}_j] & = & i\delta_{ij} \ , \\ [\tilde{p}_1, \tilde{p}_2] & = & i\tilde{B} \ . \end{array} \right. \qquad (17)$$

The associative algebra generated by $\tilde{x}_i$ and $\tilde{p}_i$ is identical to the commutative magnetic algebra and thus isomorphic to two copies of the Heisenberg algebra[b]. Therefore it has a unique irreducible representation. Any state $|f)$

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[b]This can be shown by making an additional $\tilde{x}$-dependent linear shift of the $\tilde{p}_i$ generators.

in this representation can be expanded in $\tilde{x}_i$ eigenstates

$$|f) = \int d^2y \ f(\vec{y})|\vec{y}) \ , \tag{18}$$

where the eigenstates and their relative phases are chosen to satisfy

$$\left\{ \begin{array}{rcl} \tilde{x}_i|\vec{y}) & = & y_i|\vec{y}) \ , \\ |\vec{y}) & = & e^{-i\vec{y}\cdot\tilde{p}}|\vec{0}) \ . \end{array} \right.$$

This associates a commutative wavefunction $f(\vec{y})$ with each state. Note that we will use a round bracket for states on the plane and an angle bracket for states on the torus.

The second method of obtaining the irreducible representation of the quantum algebra of observables (7) is through the x-operator representation of fields on the noncommutative plane. Define the commuting operators

$$\Delta_i = \frac{p_i + \kappa B\theta^{-1}\varepsilon_{ij}x_j}{1 + \kappa B} \ , \tag{19}$$

with κ as above. The algebra of observables for the quantum plane (7) is also generated by x_i and Δ_i which satisfy

$$\left\{ \begin{array}{rcl} [x_1, x_2] & = & i\theta \ , \\ [x_i, \Delta_j] & = & i\delta_{ij} \ , \\ [\Delta_1, \Delta_2] & = & 0 \ . \end{array} \right. \tag{20}$$

One obvious representation of the algebra defined by relations (7) is the algebra itself with the action given by the left algebra multiplication. The subalgebra generated by x_i is also a representation. To see this first note that, since Δ_i commute, we can define a state $|1)$ satisfying

$$\Delta_i|1) = 0 \ . \tag{21}$$

Then a representation is obtained by acting with arbitrary numbers of x_i and p_i on $|1)$. We can eliminate the p_i's using the inverse of (19)

$$p_i = (1 + \kappa B)\Delta_i - \kappa B\theta^{-1}\varepsilon_{ij}x_j \ ,$$

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and after commuting all the Δ_i to the right and using (21) every state can be written as $\hat{f}|1)$, where $\hat{f}$ denotes an operator constructed out of x_i's. Thus we can identify this representation with the associative algebra generated by x_i and it is convenient to drop the state $|1)$ and simply write states as $\hat{f}$. The generators act on such a state as

$$\left\{ \begin{array}{rcl} x_i(\hat{f}) & = & x_i\hat{f} \ , \\ \Delta_i(\hat{f}) & = & [\Delta_i, \hat{f}] = \theta^{-1}\, \varepsilon_{ij}[x_i, \hat{f}] \ , \end{array} \right.$$

thus x_i acts by left multiplication and Δ_i as a commutator. In the last line we first used (21), then, used the fact that in commutators we can freely substitute $\theta^{-1}\varepsilon_{ij}x_j$ for Δ_i to express the result of the action of Δ_i only in terms of x_i.

Next we find the explicit state-operator map relating the two representations. First we need the operators corresponding to the eigenstates $|\vec{y})$. After expressing the generators in (17) in terms of the generators in (20) we have

$$\tilde{x}_i(\hat{f}) = \left(1 + \frac{\kappa}{\theta}\right)\{x_i, \hat{f}\} \ .$$

The operator $\hat{f}$ corresponding to the state $|\vec{0})$ must satisfy

$$\{x_i, \hat{f}\} = 0 \ ,$$

thus it must be proportional to the parity operator P. A representation of P is [22]

$$P = \frac{1}{(2\pi)^2}\int d^2k \ e^{i\vec{k}\cdot\vec{x}} \ .$$

We can fix the overall normalization by requiring that we have the map

$$|\vec{0}) \rightarrow P \ .$$

Then we also have

$$|\vec{y}) \rightarrow e^{-i\vec{y}\cdot\tilde{p}}(P) = \frac{1}{(2\pi)^2}\int d^2k \ e^{i\vec{k}\cdot(\vec{x}-\lambda\vec{y})} \ ,$$

where

$$\lambda = \frac{(1-\theta B)^{1/2}+1}{2(1-\theta B)^{1/2}} = 1 + \frac{1}{4}\theta B + \dots \ .$$

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Finally for an arbitrary state (18) we have

$$|f) \to \hat{f} = \frac{1}{(2\pi)^2} \iint d^2y \, d^2k \; f(\vec{y}) e^{i\vec{k}\cdot(\vec{x}-\lambda\vec{y})} \; . \tag{22}$$

This is almost the standard map from commutative functions to operators for the quantum plane except for the factor λ, and it can be used to define a $*$-product on the space of commutative functions on the plane. Note that for $B = 0$ we have $\lambda = 1$ and the map reduces to the standard case. One can also check that

$$2\pi\theta \, \mathrm{Tr}(\hat{f}) = \int d^2y \; f(\vec{y}) \; ,$$

thus the inner product of commutative wave functions is mapped into

$$\left(\hat{f}, \hat{g}\right) = 2\pi\theta \, \mathrm{Tr}(\hat{f}^\dagger \hat{g}) \; .$$

5   Quantum Bundles

We now discuss how to formally obtain the representations for the torus algebra (6) by reducing the corresponding representation for the plane. The magnetic translations

$$D_i = \frac{1}{1 - \theta B}(p_i - B\varepsilon_{ij}x_j) \; , \quad i = 1, 2 \; ,$$

commute with p_i and shift the x_i in the standard way. Therefore the operators

$$V_i \equiv e^{2\pi i D_i}$$

also commute with U_i. Thus V_i generate the commutant of the torus algebra (6) in the planar algebra (7). The V_i, however, do not mutually commute but rather satisfy the 'clock and shift' algebra

$$V_1 V_2 = V_2 V_1 \, e^{2\pi i m/n} \; . \tag{23}$$

A maximal commutative subalgebra of the algebra generated by V_i is generated by the operators

$$T_i \sim V_1^{N_{i1}} V_2^{N_{i2}} \; ,$$

if $\vec{N}_1 \times \vec{N}_2 = n$. We choose the phases such that

$$T_i \equiv \exp\left(-\frac{2\pi i B}{1 - \theta B}\vec{N}_i \times \vec{x}\right)\exp\left(\frac{2\pi i}{1 - \theta B}\vec{N}_i \cdot \vec{p}\right) \ .$$

Since T_i commute with the generators of the algebra (6) we can obtain a representation of this algebra by requiring states to satisfy

$$T_i|f\rangle = e^{-i\zeta_i}|f\rangle \ . \tag{24}$$

This representation is in fact irreducible. If we suppose that it is not, there must exist an operator commuting with the generators of the algebra (6) and taking distinct eigenvalues in each irreducible subrepresentation. But then this operator must commute with T_i and, since the algebra generated by T_i is maximal, it must itself be expressed in terms of T_i. Thus it must be proportional to the identity on the whole representation. Using the constraint (24) on a state (18) implies the following quasi-periodicity

$$f(\vec{y} + 2\pi\vec{N}_i) = e^{-i(\zeta_i - \frac{m}{2n}\vec{N}_i \times \vec{y})}f(\vec{y}) \ . \tag{25}$$

Using this quasi-periodicity we can rewrite the integral over the plane as an integral over a fundamental cell of the lattice generated by $\vec{N}_i$

$$|f\rangle = \int_{\text{cell}} d^2y \ f(\vec{y})|\vec{y}\rangle \ ,$$

where

$$|\vec{y}\rangle = \sum_{k_i}(-1)^{mk_1k_2}e^{-ik_i(\zeta_i - \frac{m}{2n}\vec{N}_i \times \vec{x})}|\vec{y} + 2\pi k_i\vec{N}_i) \ .$$

Then one can check that the states $|\vec{y}\rangle$ satisfy the quasi-periodicity (14). Thus we have identified the representations of Section 3 embedded in the planar representation.

Alternatively we can obtain the same representations in the noncommutative plane operator representation by imposing

$$T_i(\hat{f}) = e^{-i\zeta_i}(\hat{f}) \ , \tag{26}$$

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on the state-operator $\hat{f}$. If we write $\hat{f}(\vec{x})$ i.e. $\hat{f}$ is an ordered "function" of x_1 and x_2 the constraint (26) implies

$$\hat{f}(\vec{x} + 2\pi \vec{N}_i) = \hat{f}(\vec{x}) \exp\left(-i\zeta_i + \frac{im}{n}(1 + \frac{\kappa}{\theta})\vec{N}_i \times \vec{x}\right) , \qquad (27)$$

These relations are exactly the defining relations of quantum bundles as discussed in [23]-[27]. Note that using the the mapping (22) from commutative functions to operators, any vector bundle with transition functions as in (25) can be mapped into a quantum bundle whose sections satisfy (27).

6 Landau Levels

We conclude by determining the structure of the energy spectrum of the particle. We can immediately see that the eigenvalues of the energy are independent of both the noncommutativity parameter and the torus periods. The Hamiltonian (3) has a harmonic oscillator structure. Defining the ladder operators $a, a^{\dagger}$

$$a = (P_1 + i \operatorname{sgn}(\mathcal{B}) P_2) / \sqrt{2|\mathcal{B}|}$$

the Hamiltonian becomes

$$H = |\mathcal{B}|(a^{\dagger}a + \frac{1}{2}) .$$

Therefore, its energy levels are of the form $E_n = |\mathcal{B}|(n + \frac{1}{2})$ which are the usual Landau level eigenvalues. (By the standard argument, there cannot be any levels in between these values, since they would violate unitarity.) It remains to determine the degeneracy of these levels. The set of all degenerate states at a given energy level can be obtained by acting on any representative state for each level with the set of all physical operators commuting with the Hamiltonian. The operators $\tilde{V}_i$ defined as

$$\begin{cases} \tilde{V}_1 & = & \exp\left(-\frac{i}{B}p_1\right) U_2 , \\ \tilde{V}_2 & = & \exp\left(-\frac{i}{B}p_2\right) U_1^{-1} , \end{cases}$$

commute with P_i and they are the minimal complete set of such operators (it can be seen that the only operator commuting with H but not the individual

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P_i is H itself.) They are the same as the operators W_i defined in section 3, but with the minimal choice of exponents $m_1 = m_2 = 1$ since we are not concerned with their commutation properties with the U_i. They can also be thought of as the operators V_i on the plane, generated by the magnetic translations, but raised to a fractional power $1/\tilde{B}$ in order to make the physical coordinate operators $U_i = e^{ix_i}$ appear. They satisfy the commutation relations

$$\tilde{V}_1 \tilde{V}_2 = e^{i\omega} \tilde{V}_2 \tilde{V}_1 \ ,$$

where the phase ω is

$$\omega = \frac{1 - \theta B}{B} = \frac{1}{\tilde{B}} = 2\pi \frac{n}{m} \ .$$

This is a 'clock and shift' algebra whose irreducible representations are $|m|$-dimensional (since m and n are relatively prime). In fact, each degenerate energy multiplet forms one such irreducible representation. Otherwise, there should exist some operator commuting with H and mixing the different irreducible components, and thus not belonging to the set generated by the $\tilde{V}_i$, which is not the case since $\tilde{V}_1, \tilde{V}_2$ generate all the commutants of H. Therefore, we conclude that the degeneracy of each Landau level is $|m|$.

The result above can be understood in terms of the density of states in each Landau level in the planar case. Using the density of states [15]

$$\rho = \frac{1}{2\pi} \left| \frac{\mathcal{B}}{1 - \Theta\mathcal{B}} \right| \ ,$$

we obtain the total number of states

$$\rho A = 2\pi |\tilde{B}| = |m/n| \ .$$

Consistency of quantization on the torus requires that there be an integer number of states per total torus area. We saw that the Hilbert space of the problem for $A\mathcal{B}/2\pi = m/n$ corresponds to quantizing on a larger torus of area nA. On that torus the number of states per Landau level is $\rho n A = |m|$.

The above results hold for $\Theta\mathcal{B} = \theta B \neq 1$. For the critical case $\mathcal{B} = \Theta^{-1}$, $\tilde{B} = \infty$, the above argument gives an infinite degeneracy of states

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per Landau level. In fact, at the critical value of $\mathcal{B}$, the representation of the physical observables reduces into the sum of an infinite number of irreducible components of the reduced algebra (9). The operators $\tilde{U}_i$ are superselected and there is nothing that could induce transitions between states $|\vec{\phi}\rangle$ with different $\vec{\phi}$. Each irreducible component is labeled by the eigenvalues of $\tilde{U}_i$ and has a unique state per Landau level.

Finally, the algebra (6) has only infinite-dimensional representations when $A\mathcal{B}/2\pi$ is irrational, and the degeneracy of each Landau level becomes infinite. The Hilbert space of the noncommutative torus in this case is the same as the one of the noncommutative plane, since there is no finite multiple of the torus containing an integer number of states per Landau level.

Acknowledgments

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